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Estimating the Global Minimum Variance Portfolio**

Abstract

According to standard portfolio theory, the tangency portfolio is the only efficient stock portfolio. However, empirical studies show that an investment in the global minimum variance portfolio often yields better out-of-sample results than does an investment in the tangency portfolio and suggest investing in the global minimum variance portfolio. But little is known about the distributions of the weights and return parameters of this portfolio. Our contribution is to determine these distributions. By doing so, we answer several important questions in asset management.

JEL-Classification: C22, G11.

Keywords: Estimation Risk; Global Minimum Variance Portfolio; Weight Estimation.

1 Introduction

Standard portfolio theory suggests that the tangency portfolio is the only efficient stock portfolio. However, many empirical studies show that an investment in the global minimum variance portfolio often yields better out-of-sample results than does an investment in the tangency portfolio (see, e.g., Jorion (1991), and Chopra and Ziemba (1993)). This result is typically attributed to the high estimation risk associated with expected returns[1]. Therefore many recent studies suggest investing into the global minimum variance portfolio instead of the tangency portfolio (see, e.g., Ledoit and Wolf (2003), and Jagannathan and Ma (2003)).

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1 In this paper we deal only with estimation risk resulting from unknown return distribution parameters. In the more general definition of Bawa et al. (1976), estimation risk also includes situations in which not only the parameters, but also the functional form of the distribution is unknown. The estimation risk with respect to expected continuous returns is highlighted by Merton (1980). Dorfleitner (2003) points out that one has to distinct clearly between discrete returns (as used in our paper) and continuous returns.

However, little is known about the distribution of the global minimum variance portfolio. Dickinson (1974) calculates the unconditional distribution of the portfolio weights in the special case of two uncorrelated assets. Okhrin and Schmid (2005) generalize this result by allowing N assets with arbitrary correlation structure. But although the conditional distribution is still unknown, it is necessary for calculating test statistics and confidence intervals in small samples.

The main contribution of our paper is to derive the conditional distribution of the estimated weights of the global minimum variance portfolio. As a by-product, we obtain estimates for the expected return and the return variance of the global minimum variance portfolio. If we know the conditional distributions, we can answer important questions in asset management. For example, what determines the extent of estimation risk? And, can an investor significantly reduce his portfolio risk by including additional assets in his portfolio?

The paper is organized as follows. In Section 2 we briefly review the traditional approach of estimating the weights of the global minimum variance portfolio. In Section 3 we present an alternative OLS estimation approach, which leads to identical weight estimates. Using this alternative estimation approach we derive in Section 4 the conditional distribution of the estimated portfolio weights and the conditional distributions of the estimated return parameters. In Section 5 we show that we can also use our OLS approach when the returns are not normally distributed. In Section 6 we apply the results of Section 4 to calculate the estimation risk associated with the estimation of the global minimum variance portfolio. We show that our weight estimator leads to the lowest estimation risk of all unbiased weight estimators. In Section 7 we give some examples of how to apply our results in international asset management. Section 8 concludes.

2 Traditional Approach

Assume that there are N stocks in the capital market. We denote the discrete return of stock i from year $t-1$ to year t by $r_{t,i}$. The vector μ contains the expected returns per year, μ_i, of the N stocks. The $N \text{x} N$ matrix Σ contains the return variances and covariances σ_{ij}. We assume that the returns are multivariate normally distributed. In addition, they are identically and independently distributed.

The global minimum variance portfolio (MV) is the stock portfolio with the lowest return variance for a given covariance matrix Σ . It is the solution to the following minimization problem:

$$\min_{w=(w_1,\ldots,w_N)} w'\Sigma\, w \quad s.t.\ w'\,\underline{1} = 1 \tag{1}$$

where $\underline{1}$ is a column vector of appropriate dimension whose entries are ones and $w = (w_1,\ldots,w_N)'$ is a vector of portfolio weights. The weights $w_{MV} = (w_{MV,1},\ldots,w_{MV,N})'$ of the global minimum variance portfolio are given as

$$w_{MV} = \frac{\Sigma^{-1}\underline{1}}{\underline{1}`\Sigma^{-1}\underline{1}}. \tag{2}$$

The expected return μ_{MV} and the return variance σ^2_{MV} of the global minimum variance portfolio are given as

$$\mu_{MV} = \mu`w_{MV} = \frac{\mu`\Sigma^{-1}\underline{1}}{\underline{1}`\Sigma^{-1}\underline{1}} \tag{3}$$

and

$$\sigma^2_{MV} = w`_{MV}\Sigma w_{MV} = \frac{1}{\underline{1}`\Sigma^{-1}\underline{1}}. \tag{4}$$

The lower variance bound σ^2_{MV} can only be attained if we know the covariance matrix Σ of the stock returns. As highlighted before, we do not know the covariance matrix Σ. We must estimate it in real markets. Typically, researchers use historical return observations for this estimation.

The traditional estimation approach is to replace the expected returns μ and the covariance matrix Σ with their maximum likelihood estimators $\hat{\mu}$ and $\hat{\Sigma}$. in Equations (2)-(4). The estimated portfolio weights $\hat{w}_{MV}$ and return parameters $\hat{\mu}_{MV}$ and $\hat{\sigma}^2_{MV}$ of the global minimum variance portfolio are nonlinear functions of the stock return parameter estimates $\hat{\mu}$ and $\hat{\Sigma}$. . Therefore, the distributions of $\hat{w}_{MV}$, $\hat{\mu}_{MV}$ and $\hat{\sigma}^2_{MV}$ are hard to determine, even if we do know the distributions of the parameter estimates $\hat{\mu}$ and $\hat{\Sigma}$.

3 OLS Approach

We use a regression-based approach to determine the weights w_{MV}, the expected return μ_{MV} and the return variance σ^2_{MV} of the global minimum variance portfolio. We rewrite the weights of the global minimum variance portfolio as regression coefficients. Without loss of generality, we choose the return of stock N to be the dependent variable:

$$r_{t,N} = \alpha + \beta_1(r_{t,N} - r_{t,1}) + \ldots + \beta_{N-1}(r_{t,N} - r_{t,N-1}) + \varepsilon_t \qquad t = 1,\ldots,T \tag{5}$$

where ε_t is a noise term that satisfies the standard assumptions of the classical linear regression model regarding errors. We note that the error term ε_t is by construction uncorrelated with all the return differences $r_{t,N} - r_{t,i}$. This absence of correlation allows us to apply the OLS estimation technique. To make use of the usual test statistics, we must also stipulate independence.

The three statements in Proposition 1 describe the relation between the linear regression and the global minimum variance portfolio.

Proposition 1

1. The regression coefficients $\beta_1,\ldots,\beta_{N-1}$ in Equation (5) correspond to the portfolio weights $w_{MV,1},\ldots, w_{MV,N-1}$ of the global minimum variance portfolio:

$$\beta_i = w_{MV,i}. \tag{6}$$

2. The coefficient α in Equation (5) corresponds to the expected return μ_{MV} of the global minimum variance portfolio:

$$\alpha = \mu_{MV}. \tag{7}$$

3. The variance of the noise term in Equation (5) corresponds to the variance of the global minimum variance portfolio:

$$\sigma_\varepsilon^2 = \sigma_{MV}^2. \tag{8}$$

To prove this proposition, we define β^{ex}, w_{MV}^{ex} and r_t^{ex} as column vectors of dimension $N-1$. The superscript *ex* indicates that the vector has no entry for asset N, i.e. these vectors contain the entries β_i^{ex}, $w_{MV,i}^{ex}$ and $r_{t,i}^{ex}$ with $i = 1,\ldots,N-1$. The $(N-1)$x$(N-1)$ matrix Ω is the covariance matrix of the regressors of Equation (5):

$$\Omega := \text{var}(r_{t,N}\underline{1} - r_t^{ex}). \tag{9}$$

The regression coefficients β^{ex} are the standardized covariances of the regressors and the dependent variable:

$$\beta^{ex} = \Omega^{-1}\ \text{cov}\,(r_{t,N}\underline{1} - r_t^{ex}, r_{t,N}). \tag{10}$$

We must show that the weights w_{MV}^{ex} of the global minimum variance portfolio correspond to the regression coefficients β^{ex}. We can then compute the weight $w_{MV,N}$ as $1-(w_{MV}^{ex})'\underline{1}$.

To prove $\beta^{ex} = w_{MV}^{ex}$, we consider an arbitrary portfolio P. Its return is determined by the weight vector $w_P^{ex} = (w_{P,1},\ldots, w_{P,N-1})'$ and the stock returns r_t^{ex} and $r_{t,N}$:

$$r_{t,P} = (w_P^{ex})' r_t^{ex} + (1 - (w_P^{ex})'\underline{1})\, r_{t,N} = r_{t,N} - (w_P^{ex})'(r_{t,N}\underline{1} - r_t^{ex}) \tag{11}$$

The return variance of this arbitrary portfolio P

$$\sigma_P^2 = \sigma_N^2 + (w_P^{ex})'\Omega w_P^{ex} - 2(w_P^{ex})'\text{cov}(r_{t,N}\underline{1} - r_t^{ex}, r_{t,N}) \tag{12}$$

is a function of the weights w_P^{ex}. To find the weights of the global minimum variance portfolio, we minimize Equation (12) with respect to the portfolio weights w_P^{ex}. This minimization leads to

$$w_{MV}^{ex} = \Omega^{-1}\text{cov}(r_{t,N}\underline{1} - r_t^{ex}, r_{t,N}). \tag{13}$$

The weights (13) correspond to the regression coefficients (10), which proves the first statement of Proposition 1.

To prove Statements 2 and 3, we rearrange Equation (5) and use $\beta_i = w_{MV,i}$:

$$\alpha + \varepsilon_t = w_{MV,1} \cdot r_{t,1} + \ldots + w_{MV,N-1} \cdot r_{t,N-1} + \left(1 - \sum_{i=1}^{N-1} w_{WM,i}\right) \cdot r_{t,N}. \quad (14)$$

The right-hand side of Equation (14) is the return of the global minimum variance portfolio. Applying the expectation and the variance operator to Equation (14) proves Statements 2 and 3.

Proposition 1 shows that both the traditional and the OLS approaches lead to identical portfolio weights. However, the result is based on the assumption of known parameters. We show that the identity result holds even if we have to estimate the parameters. We define the OLS estimates of the coefficients in Equation (5) as $\hat{\alpha}, \hat{\beta}_1, \ldots, \hat{\beta}_{N-1}$. $\hat{\sigma}_\varepsilon^2 = \frac{1}{T-N} {}_{t=1}^{T} \hat{\varepsilon}_t^2$ is the OLS estimate of the variance of ε_t.

Proposition 2

1. The traditional weight estimate $\hat{w}_{MV,i}$ equals the OLS estimate:

$$\hat{w}_{MV,i} = \hat{\beta}_i \quad \forall i = 1,\ldots,N-1 \quad (15)$$

$$\hat{w}_{MV,N} = 1 - \sum_{i=1}^{N-1} \hat{\beta}_i. \quad (16)$$

2. The traditional estimate of the expected return $\hat{\mu}_{MV}$ of the global minimum variance portfolio equals the OLS estimate:

$$\hat{\mu}_{MV} = \hat{\alpha}. \quad (17)$$

3. The traditional estimate of the return variance of the global minimum variance portfolio $\hat{\sigma}_{MV}^2$ is a multiple of the OLS estimate of the variance $\hat{\sigma}_\varepsilon^2$:

$$\hat{\sigma}_{MV}^2 = \frac{T-N}{T} \hat{\sigma}_\varepsilon^2. \quad (18)$$

First, we prove Statement 1. The traditional approach is the solution to the minimization problem

$$\min_{w_1,\ldots,w_N} \sum_{j=1}^{N} \sum_{i=1}^{N} w_i w_j \hat{\sigma}_{ij}. \quad (19)$$

In the OLS approach, we estimate the regression coefficients by solving the following minimization problem

$$\min_{\alpha, \beta_1, \ldots, \beta_{N-1}} \sum_{t=1}^{T} \varepsilon_t^2. \tag{20}$$

(20) can be rewritten as

$$\min_{\alpha, \beta_1, \ldots, \beta_{N-1}} \sum_{t=1}^{T} \left[-\alpha + \beta_1 r_{t,1} + \ldots + \beta_{N-1} r_{t,N-1} \left(1 - \sum_{i=1}^{N-1} \beta_i \right) r_{t,N} \right]^2. \tag{21}$$

Since the coefficients β_i correspond to the portfolio weights w_i (Proposition 1), and since the N portfolio weights add up to one, we can rearrange Equation (21) as follows:

$$\min_{\alpha, w_1, \ldots, w_N} [-\alpha + w_1 r_{t,1} + \ldots + w_{N-1} r_{t,N-1}]^2 \quad s.t. \sum_{i=1}^{N} w_i = 1 \tag{22}$$

Differentiating (22) with respect to α leads to the necessary condition for a minimum:

$$\alpha = w_1 \hat{\mu}_1 + \ldots + w_N \hat{\mu}_N. \tag{23}$$

Here, $\hat{\mu}_i = \frac{1}{T} \sum_{t=1}^{T} r_{t,i}$ is the estimated mean return of asset i. Using (23), we rewrite (22) as

$$\min_{w_1, \ldots, w_N} \sum_{t=1}^{N} \left[w_1 (r_{t,1} - \hat{\mu}_1) + \ldots + w_N (r_{t,N} - \hat{\mu}_N) \right]^2 \tag{24}$$

subject to the condition that the N portfolio weights add up to one.

Rearranging the sum in (24) yields another representation of the OLS approach (20):

$$\min_{w_1, \ldots, w_N} \sum_{i=1}^{N} \sum_{j=1}^{N} \left[w_i w_j \sum_{t=1}^{T} (r_{t,i} - \hat{\mu}_i)(r_{t,i} - \hat{\mu}_j) \right] = \min_{w_1, \ldots, w_N} \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \hat{\sigma}_{ij}. \tag{25}$$

Thus, the sum of the squared residuals in (20) is equivalent to (25). Since (25) and (19) differ only by the positive factor T, both optimization problems produce the same portfolio weights. These remarks prove the first statement of Proposition 2.

We derive Statement 2 from the necessary condition (23). Replacing w_i by $\hat{w}_{MV,i}$ makes $\hat{\alpha}$ the estimated expected return of the global minimum variance portfolio, which leads to $\hat{\alpha} = \hat{\mu}' \hat{w}_{MV}$. The expression $\hat{\mu}' \hat{w}_{MV}$ equals the traditional estimator $\hat{\mu}_{MV}$.

We derive Statement 3 accordingly. The sum of the squared residuals (20) equals $T\hat{\sigma}_{MV}^2$. This result can easily be seen by rewriting (25) as $T \min_w w'\Sigma w$. Its solution $T\hat{w}'_{MV}\Sigma\hat{w}_{MV}$ equals T times the estimated variance of the global minimum variance portfolio.

Proposition 2 states that the OLS estimation technique and the traditional approach yield identical estimates of the portfolio weights of the global minimum variance portfolio. Therefore, the estimates of $\hat{w}_{MV,i}$ are identical. The variance estimates differ only by the scalar $(T - N) / T$.

By using the equivalence of the two estimation approaches, we can transfer all the distributional results of the OLS approach to the traditional approach. Thus, we have a powerful yet simple way of deriving the conditional distributions of the estimated weights and return parameters.

4 Conditional Distribution

We estimate the weights of the global minimum variance portfolio by using the linear regression (5). As before, we assume serially independent and normally distributed returns. We define the matrix Z, which contains the regressors $z_t = (r_{t,N} - r_{t,1}, \ldots, r_{t,N} - r_{t,N-1})'$ of the linear regression (5):

$$Z := \begin{pmatrix} 1 & z_1' \\ \vdots & \vdots \\ 1 & z_T' \end{pmatrix} = (\underline{1} \; z) \tag{26}$$

The vector $\bar{z} = \frac{1}{T}\sum_{t=1}^{T} z_t$ consists of the arithmetic averages of the regressors.

Proposition 3 gives the conditional distributions of the estimated portfolio weights and return parameters. The information set we condition on consists of the T x $(N - 1)$matrix of return differences z.

Proposition 3

1. The OLS estimates of the portfolio weights, $\hat{\beta}^{ex}$, are jointly normally distributed:

$$\hat{\beta}^{ex} | z \sim N\left(w_{MV}^{ex}; \sigma_{MV}^2 \left(z'z - T\,\bar{z}\bar{z}'\right)^{-1}\right). \tag{27}$$

2. The OLS estimate of the expected return, $\hat{\alpha}$, is normally distributed:

$$\hat{\alpha} | z \sim N\left(\mu_{MV}; \sigma_{MV}^2 \left(1/T + \bar{z}'(z'z - T\bar{z}\bar{z}')^{-1}\bar{z}\right)\right). \tag{28}$$

3. Let σ_ε^2 be the OLS estimate of the variance of the error term ε_t. The following expression is χ^2-distributed:

$$(T-N)\frac{\hat{\sigma}_{\varepsilon}^{2}}{\sigma_{MV}^{2}} \sim \chi^2\,(T-N). \tag{29}$$

Proposition 3 is based on Proposition 1. The OLS estimator $\hat{B} = (\hat{\alpha}, \hat{\beta}_1, \ldots, \hat{\beta}_{N-1})' = (Z'Z)^{-1}Z'r_N$ with $r_N = (r_{1,N}, \ldots, r_{T,N})'$ is normally distributed:

$$\hat{B}\,|\,z \sim N(B;\ \sigma_{MV}^2 (Z'Z)^{-1}). \tag{30}$$

$B = (\alpha, \beta_1, \ldots, \beta_{N-1})'$ is the parameter vector. From (30), we see that the expectations of the conditional estimators $\hat{\beta}^{ex}$ and $\hat{\alpha}$ are β^{ex} and α. According to Proposition 1, the variance σ_{ε}^2 is equal to the variance of the global minimum variance portfolio σ_{MV}^2.

Using (26), we partition the matrix $Z'Z$:

$$Z'Z = \begin{pmatrix} T & T\bar{z}' \\ T\bar{z} & z'z \end{pmatrix}. \tag{31}$$

The inversion of the matrix $Z'Z$ yields[2]:

$$(Z'Z)^{-1} = \begin{pmatrix} 1/T + \bar{z}'(z'z - T\bar{z}\bar{z}')^{-1}\bar{z} & \bar{z}'(z'z - T\bar{z}\bar{z}') \\ (z'z - T\bar{z}\bar{z}')^{-1}\bar{z} & (z'z - T\bar{z}\bar{z}')^{-1} \end{pmatrix}. \tag{32}$$

σ_{MV}^2 times the upper left element of the right hand side of Equation (32) is the conditional variance of $\hat{\alpha}$. σ_{MV}^2 times the lower right element is the conditional covariance matrix of $\hat{\beta}^{ex}$.

Proposition 3 states the core results of this paper. It allows us to calculate the estimation risk involved in estimating the global minimum variance portfolio, and to carry out statistical tests concerning the estimated weights and return parameters.

5 Non-Normal Returns

Throughout this paper, we assume that stock returns are independent and normally distributed. However, we can also use our OLS approach for non-normal returns.

Instead of restricting ourselves to the multivariate normal distribution, we now consider the broader class of elliptical distributions. Among others, the class of elliptical distributions comprises the normal distribution and the Student-t-distribution. We choose this class of distributions for two reasons. First, because elliptical distributions support mean variance analysis since they fulfil the two requirements: they can be entirely characterized by their

2 See Greene (2000, 34).

mean and variance, and linear combinations of elliptically distributed random variables are again elliptically distributed. Second, because elliptical distributions can describe empirical features of stock returns, especially the heavy tails of stock return distributions[3].

If we no longer assume normally distributed returns, but instead assume elliptically distributed returns, then the noise term ε_t in Equation (5) will remain uncorrelated of the regressors z_t. However, the error term will not necessarily be independent of the regressors. For instance, the correlation of the squared noise term ε_t^2 and the squared regressors $z_{t,i}^2$ may be different from zero. This dependence means that the standard assumptions of the linear regression model are no longer fulfilled, because the noise terms ε_t are heteroskedastic, i.e., the variance of ε_t varies in a systematic way. Nevertheless, we can apply the OLS method. Propositions 1 and 2 remain unaltered, but we have to modify the crucial Proposition 3. The OLS estimates remain unbiased and consistent, but the estimates $\hat{\alpha}$ and $\hat{\beta}$ are only asymptotically normally distributed. We must also modify the estimated covariance matrix. To obtain correct standard errors in the regression, we can adjust the covariance matrix using White's (1980) correction. If there is not only heteroskedasticity, but also autocorrelation in the data, we must use the Newey and West (1987) correction instead.

6 Estimation Risk

Again assuming independent and normally distributed returns, we analyze the quality of the traditional weight estimates. We judge the quality of the estimator by looking at the estimation risk. The estimation risk is the additional out-of-sample return variance due to errors in the estimated portfolio weights.

We consider an investor who uses T return observations $r_1,\ldots,r_T$ to estimate $\hat{w}_{MV}$. Using the estimates $\hat{w}_{MV}$, the investor invests his funds for the period to follow. This strategy yields the out-of-sample return $\hat{r}_{T+1,MV} = \hat{w}_{MV}' r_{T+1}$. Its risk is $\mathrm{var}(\hat{r}_{T+1,MV} | r_1,\ldots,r_T)$ which depends on the realizations of the stock returns from $t = 1$ to $t = T$.

Proposition 4
If the portfolio weights are estimated traditionally, then the conditional out-of-sample return variance is given by

$$\mathrm{var}(\hat{r}_{T+1,MV} | r_1,\ldots,r_T) = \sigma_{MV}^2 + \tilde{R}(\hat{w}_{MV}) \tag{33}$$

with

3 There is another feature of normally distributed returns that is not in accordance with stock returns: due to the limited liability, stock returns are bounded by –1. Under the assumption of the normal distribution, there is a strictly positive probability for returns to be smaller than –1. This feature characterizes other distributions within the class of elliptical distributions. However, there are elliptical distributions with bounded support. See Ingersoll (1987, 105).

$$\tilde{R}(\hat{w}_{MV}) = (\hat{w}_{MV} - w_{MV})'\Sigma(\hat{w}_{MV} - w_{MV}). \tag{34}$$

Proposition 4 (proof in Appendix 1) shows that the risk depends on two components. The first component, σ^2_{MV}, is the innovation risk, i.e., the risk due to the randomness of stock returns. The second component, $\tilde{R}(\hat{w}_{MV})$, is the estimation risk. If the investors knew all return distribution parameters, they would choose (2) as their weights when they select the global minimum variance portfolio. In this case, there is no estimation risk and (33) reduces to (4). However, since the investor does not know the distribution parameters and instead must estimate them, his estimated portfolio weights, $\hat{w}_{MV}$, differ from the true weights, w_{MV}. This difference leads to the conditional estimation risk $\tilde{R}(\hat{w}_{MV})$. We note that the w_{MV} is a random variable that takes on only positive values or zero. The more the estimated weights differ from the true weights, the larger is the estimation risk. We obtain the unconditional estimation risk by applying the expectation operator to var($\hat{r}_{T+1,MV}| r_1, \ldots, r_T$).

Proposition 5

If the portfolio weights are estimated traditionally, then the unconditional out-of-sample return variance is given by

$$E(\text{var}(\hat{r}_{T+1,MV}| r_1,\ldots,r_T)) = \sigma^2_{MV} + \bar{R}(\hat{w}_{MV}) \tag{35}$$

with

$$\bar{R}(\hat{w}_{MV}) = \sigma^2_{MV}\frac{N-1}{T-N-1}. \tag{36}$$

According to this proposition (proof in Appendix 2) the larger the innovation risk σ^2_{MV}, the larger the investment universe N and the shorter the estimation period T, the higher is the unconditional estimation risk $\bar{R}(\hat{w}_{MV})$. This result supports the claim in Jagannathan and Ma (2003), according to which the estimation risk is most prominent in large investment universes.

Proposition 6 proves that the estimation risk cannot be reduced by choosing another unbiased weight estimator. The traditional weight estimator is the best unbiased estimator.

Proposition 6

The traditional weight estimator $\hat{w}_{MV}$ given in Equation (15) has the lowest unconditional estimation risk $\bar{R}(\cdot)$ of all unbiased weight estimators $\breve{w}_{MV}$

$$\bar{R}(\hat{w}_{MV}) \leq \bar{R}(\breve{w}_{MV}) \tag{37}$$

This proposition follows from the properties of OLS estimators. When there are independent and normally distributed error terms, the OLS estimator is the best unbiased weight estimator. According to Proposition 2, this statement is also true for the traditional estimator. In Appendix 3 we show that this property implies the lowest estimation risk possible.

Although Proposition 4 still holds without the normality assumption, Proposition 5 is no longer valid. But if there is considerable estimation risk even in the best of all situations (iid-returns, normal distribution), then there is reason to believe that there will be a lot more estimation risk in less favourable situations. When there are non-normal returns, we must modify Proposition 6. In this case, the traditional estimator is only the best *linear* unbiased estimator.

7 Statistical Inference

Here, we use our distributional results to address problems in international asset allocation. We conduct an empirical study based on international stock data. Our data set consists of monthly MSCI total return indexes of the G7 countries: Canada, France, Germany, Italy, Japan, the United Kingdom, and the United States. These countries cover the major currency regions (dollar, euro, pound, yen). We calculate all indexes in euros, i.e., we take the view of an German investor. The data set covers the period from January 1984 to December 2003.

We choose the return of the German index as the dependent variable $r_{t,N}$ in the regression (5). We run the regression and obtain estimates of the portfolio weights of the global minimum variance portfolio. In *Table 1* we report the weight estimates $\hat{w}_{MV;i}$, their standard errors and the *t*-statistics. Appendix 4 provides the exact formula of the test statistic. As the stock returns show more kurtosis than is compatible with the normal distribution, we apply the White (1980) correction to obtain correct standard errors.

Table 1: Weight Estimates of the Global Minimum Variance Portfolio

Country (*i*)	Weight	Standard Error	*t*-Statistic
Canada (*Can*)	0.0146	0.0998	0.1467
France (*Fra*)	0.0756	0.0772	0.9799
Germany (*Ger*)	0.1418	0.0881	1.6088
Italy (*Ita*)	0.0427	0.0512	0.8336
Japan (*Jap*)	0.1909	0.0570	3.3512
United Kingdom (*UK*)	0.3536	0.0946	3.7362
United States (*USA*)	0.1807	0.1067	1.6947

We estimate the weights of the global minimum variance portfolio for a universe of seven countries, represented by their MSCI equity market indexes (euro based, total return indexes, monthly observations, 1984-2003). We use a regression-based approach for the weight estimation and we apply the White-correction for the standard errors.

Table 1 highlights that the UK market has the highest weight in the international global minimum variance portfolio, followed by Japan and the U.S. Only the weights for the indexes of Japan, the UK and the U.S. are significantly different from zero at the 10% level. This empirical result suggests that a German investor who holds only German stocks should add American, Japanese, and British stocks to his domestic holdings.

To test whether a German investor can exclude several countries from his portfolio without increasing the risk of his portfolio, we apply the *F*-test as shown in Appendix 4. By using the *F*-test we can simultaneously test several linear restrictions on the portfolio weights. Our test is a simplified version of a spanning test. The spanning tests suggested in the literature (see, e.g., Kan and Zhou (2001)) test whether the inclusion of an additional asset changes the minimum variance frontier. Our test focuses not on the whole frontier, but on only one portfolio of the frontier, the global minimum variance portfolio. If we find a significant change in the global minimum variance portfolio, then we know that the minimum variance frontier has also changed. Thus, our test is sufficient for spanning. Since the global minimum variance portfolio does not depend on expected returns, we expect our test to have a higher power than traditional spanning tests[4].

First, we want to know whether a German investor can reduce his portfolio risk by diversifying internationally. We test the hypothesis:

$H_{0,1}$ International diversification does not pay for German investors, i.e. $w_{MV,UK} = w_{MV,USA} = w_{MV,Fra} = w_{MV,Jap} = w_{MV,Ita} = w_{MV,Can} = 0$

The null hypothesis is rejected at the 1%-level ($F(6,233)$-statistic = 22.45). Thus, it pays for a German investor to diversify internationally.

Next, we analyze whether adopting a naive diversification strategy or diversifying optimally makes a difference:

$H_{0,2}$ Naive diversification ($w_{MV,i} = 1/7 \ \forall \ i$) offers the same risk diversification effect as optimal diversification.

$H_{0,2}$ is rejected at the 10% level ($F(6,233)$-statistic = 2.06). We conclude that a German investor is better off choosing the weights according to (2) than by investing equally in all countries.

Finally, we want to know whether investing in only *one* country per currency region reduces the diversification effect significantly. The countries invested in are Germany (euro), Japan (yen), the UK (pound) and the United States (dollar).

$H_{0,3}$ Investing in one country per currency region ($w_{MV,Can} = w_{MV,Fra} = w_{MV,Ita} = 0$) offers the same risk diversification as investing in all countries.

4 Jorion (1985) develops an alternative test to address this question. He uses a maximum likelihood test to check whether a given portfolio is significantly different from the global minimum variance portfolio. Although the distribution of the Jorion test is known only asymptotically, the distribution of our test is known even in small samples.

We cannot reject $H_{0,3}$ ($F(3{,}233)$-statistic $= 0.58$). The results suggest that covering the major currency regions by choosing only one country for each currency region provides sufficient diversification.

The three hypotheses tested above are examples of how to use the results of this paper. Obviously, it is easy to find other hypotheses to test with our approach.

8 Conclusion

In this paper we show that the weights of the global minimum variance portfolio are equal to regression coefficients. This finding allows us to transfer the entire OLS methodology to the estimation of the weights and return parameters of the global minimum variance portfolio. Using the OLS methodology, we can derive the conditional distributions of the estimated portfolio weights and estimated return parameters. These conditional distributions are necessary for analyzing the global minimum variance portfolio, and they are a contribution to the literature on distributions of estimated portfolio weights.

We discuss two applications of our distributional results. The first application is to assess the extent of the estimation risk involved in estimating the global minimum variance portfolio. Our second application is to test important hypotheses in international asset management. These two applications serve as an illustration of the usefulness of our approach.

Appendix 1

Using $\hat{w}_{MV} = w_{MV} + (\hat{w}_{MV} - w_{MV})$ we rewrite the conditional out-of-sample return variance as

$$\begin{aligned} \text{var}(\hat{r}_{T+1,MV} | r_1,\ldots,r_T) &= \hat{w}'_{MV}\Sigma\hat{w}_{MV} \\ &= \sigma^2_{MV} + (\hat{w}_{MV} - w_{MV})'\Sigma(\hat{w}_{MV} - w_{MV}) \\ &\quad + 2w'_{MV}\Sigma(\hat{w}_{MV} - w_{MV}). \end{aligned} \tag{38}$$

We can rewrite the last term in as

$$2(w'_{MV}\Sigma\hat{w}_{MV} - w'_{MV}\Sigma w_{MV}). \tag{39}$$

The first term is the return covariance of a portfolio with the portfolio weights $\hat{w}_{MV}$ and the global minimum variance portfolio w_{MV}. The second term is the return variance of the global minimum variance portfolio. Huang and Litzenberger (1988, 68) prove that the return covariance of an arbitrary stock portfolio and the global minimum variance portfolio is equal to the return variance of the global minimum variance portfolio. Therefore, the last term in (38) drops out. We have now completed the proof of Proposition 4.

Appendix 2

Here, we prove Proposition 5.

Lemma 1 shows how to express the unconditional estimation risk $R(\cdot)$ of any unbiased weight estimator $\breve{w}_{MV}$ as a function of the estimator's unconditional variance $\text{var}(\breve{w}^{ex}_{MV})$.In Lemma 2 we compute the unconditional variance of a specific unbiased weight estimator, the traditional weight estimator. By combining these two lemmata, we obtain the expression for the estimation risk $\overline{R}(\hat{w}_{MV})$ as stated in Proposition 5.

Lemma 1
Let $\breve{w}_{MV}$ be any unbiased weight estimate. Then the unconditional out-of-sample return variance is

$$E(\text{var}(\breve{r}_{T+1,MV}| r_1,\ldots,r_T)) = \sigma^2_{MV} + \overline{R}(\breve{w}_{MV}) \tag{40}$$

with

$$\overline{R}(\breve{w}_{MV}) = tr[\text{var}(\breve{w}^{ex}_{MV})\Omega]. \tag{41}$$

Proof of Lemma 1: Using (11) we can rewrite the out-of-sample return as

$$\breve{r}_{T+1,MV} = r_{T+1,N} - (\breve{w}^{ex}_{MV})'(r_{T+1,N}\,\underline{1} - r^{ex}_{T+1}). \tag{42}$$

The unconditional out-of-sample variance is

$$E(\text{var}(\breve{r}_{T+1,MV}| r_1,\ldots,r_T)) = \sigma^2_N + E((\breve{w}^{ex}_{MV})'\Omega(\breve{w}^{ex}_{MV})) - 2E(\breve{w}^{ex}_{MV})'\text{cov}(r_{T+1,N}\underline{1} - r^{ex}_{T+1}, r_{T+1,N}). \tag{43}$$

Setting $E(\breve{w}^{ex}_{MV}) = w^{ex}_{MV} + E(\breve{w}^{ex}_{MV} - w^{ex}_{MV})$, we rewrite the expression $E((\breve{w}^{ex}_{MV})'\Omega\,\breve{w}^{ex}_{MV})$ as

$$E\Big((\breve{w}^{ex}_{MV})'\Omega\,\breve{w}^{ex}_{MV}\Big) = (w^{ex}_{MV})'\Omega w^{ex}_{MV} + E\big((\breve{w}^{ex}_{MV} - w^{ex}_{MV})'\Omega(\breve{w}^{ex}_{MV} - w^{ex}_{MV})\big) + 2E(\breve{w}^{ex}_{MV} - w^{ex}_{MV})'\Omega w^{ex}_{MV}. \tag{44}$$

Inserting (44) in (43) and using

$$\sigma^2_{MV} = \sigma^2_N + (w^{ex}_{MV})'\Omega\,\breve{w}^{ex}_{MV} - 2(w^{ex}_{MV}))'\text{cov}(r_{T+1,N}\,\underline{1} - r^{ex}_{T+1}, r_{T+1,N}) \tag{45}$$

we get

$$E\Big(\text{var}(\breve{r}_{T+1,MV}| r_1,\ldots,r_T)\Big) = \sigma^2_{MV} + E\Big((\breve{w}^{ex}_{MV} - w^{ex}_{MV})'\Omega(\breve{w}^{ex}_{MV} - w^{ex}_{MV})\Big). \tag{46}$$

Finally, we deal with the expression $E\Big((\breve{w}^{ex}_{MV} - w^{ex}_{MV})'\Omega(\breve{w}^{ex}_{MV} - w^{ex}_{MV})\Big)$.

$$
\begin{aligned}
E((\breve{w}_{MV}^{ex} - w_{MV}^{ex})'\Omega(\breve{w}_{MV}^{ex} - w_{MV}^{ex})) &= E(tr((\breve{w}_{MV}^{ex} - w_{MV}^{ex})'\Omega(\breve{w}_{MV}^{ex} - w_{MV}^{ex}))) \\
&= E(tr((\breve{w}_{MV}^{ex} - w_{MV}^{ex})(\breve{w}_{MV}^{ex} - w_{MV}^{ex})'\Omega)) \\
&= tr((\breve{w}_{MV}^{ex} - w_{MV}^{ex})(\breve{w}_{MV}^{ex} - w_{MV}^{ex})')\Omega) \\
&= tr(\mathrm{var}(\breve{w}_{MV}^{ex})\Omega) \qquad (47)
\end{aligned}
$$

Lemma 1 results directly from (46) in combination with (47).

The estimation risk given by (41) depends on the estimator's variance $\mathrm{var}(\breve{w}_{MV}^{ex})$. For the traditional estimator we can state this variance explicitly. We do this in Lemma 2.

Lemma 2
The unconditional variance of the traditional weight estimator ($\hat{w}_{MV}^{ex}$) is

$$\mathrm{var}(\hat{w}_{MV}^{ex}) = \sigma_{MV}^2 \frac{1}{T-N-1}\Omega^{-1}. \qquad (48)$$

Proof of Lemma 2: From the first statement of Proposition 2 in connection with the first statement of Proposition 3 we get the conditional variance:

$$\mathrm{var}(\hat{w}_{MV}^{ex}|\, z) = \sigma_{MV}^2\,(z'z - T\,\bar{z}\bar{z}')^{-1}. \qquad (49)$$

The variance decomposition theorem provides the relation between the unconditional and conditional variance:

$$\mathrm{var}(\hat{w}_{MV}^{ex}) = E(\mathrm{var}(\hat{w}_{MV}^{ex}|\, z)) + \mathrm{var}(E(\hat{w}_{MV}^{ex}|\, z)). \qquad (50)$$

Since the estimator $\hat{w}_{MV}^{ex}$ is unbiased, the second term on the right hand side of (50) is zero. Therefore, it remains to determine the expectation of $(z'z - T\,\bar{z}\bar{z}')^{-1}$. The matrix $(z'z - T\,\bar{z}\bar{z}')$ is Wishart distributed, which follows from the assumption of normally distributed returns:

$$z'z - T\,\bar{z}\bar{z}' = \sum_{t=1}^{T}\,(z_t - \bar{z})(z_t - \bar{z})' \sim W(\Omega,\, T-1,\, N-1). \qquad (51)$$

The expectation of a random matrix whose inverse is Wishart distributed is shown in Press (1972, 112):

$$E((z'z - T\,\bar{z}\bar{z}')^{-1}) = \frac{1}{T-N-1}\Omega^{-1}. \qquad (52)$$

Lemma 2 follows immediately from (52). Inserting (48) into (41) yields (36). We have now completed the proof of Proposition 5.

Appendix 3

Based on (40) of Lemma 1, we can state the difference in the unconditional estimation risk between using an arbitrary unbiased weight estimator $\breve{w}_{MV}$ and using the traditional estimator $\hat{w}_{MV}$, respectively:

$$\overline{R}(\breve{w}_{MV}) - \overline{R}(\hat{w}_{MV}) = tr[\text{var}(\breve{w}^{ex}_{MV})\Omega] - tr[\text{var}(\hat{w}^{ex}_{MV})\Omega] = tr[\Delta\Omega] \tag{53}$$

with

$$\Delta = \text{var}(\breve{w}^{ex}_{MV}) - \text{var}(\hat{w}^{ex}_{MV}). \tag{54}$$

As $\hat{w}^{ex}_{MV}$ is the *best unbiased estimator*, the difference matrix Δ is at least positive semi-definite. Since the trace of the matrix product of two semi-definite matrices is never negative, the expression $tr[\Delta\Omega]$in (53) is not negative, either[5]. Therefore, there is no unbiased weight estimator with lower unconditional estimation risk than that of the traditional estimator[6].

Appendix 4

Here, we provide the test statistics used in Section 7. In the case of non-normally distributed returns, these statistics are only asymptotically exact and we must adjust the estimated covariance matrix as noted in Section 5.

Let $q = (q_1,\dots,q_{N-1})'$ be an arbitrary non-stochastic vector. Then the following statistic is *t*-distributed:

$$\frac{q'\hat{w}^{ex}_{MV} - q' w^{ex}_{MV}}{\sqrt{\hat{\sigma}^2_{\varepsilon}\, q'(z'z - T\,\bar{z}\bar{z}')^{-1} q}} \sim t(T-N) \tag{55}$$

Since the estimated weight of asset N is a linear combination of the other weights, i.e., $\hat{w}_{MV,N} = 1 - \underline{1}'\hat{w}^{ex}_{MV}$, we can derive the distribution of $\hat{w}_{MV,N}$ from (55) by setting $q = -\underline{1}$:

$$\frac{w_{MV,N} - \hat{w}_{MV,N}}{\sqrt{\hat{\sigma}^2_{\varepsilon}\, \underline{1}'(z'z - T\,\bar{z}\bar{z}')^{-1}\underline{1}}} \sim t(T-N). \tag{56}$$

In the third column of *Table 1*, we report the *t*-statistic as computed by for the weights $i = Can, Fra, Ita, Jap, Uk, US$ and by (56) for the weight $i = Ger$.

5 See Lütkepohl (1996, 21).

6 If we give up the assumption of normality, the traditional estimator is the best linear unbiased estimator. For the Gauss-Markov-Theorem see, e.g., Hayashi (2000, 27-29).

Let SSR and SSR_R be the sum of the squared residuals in the unrestricted and restricted regression. Let $m \leq N-1$ be the number of linear independent restrictions. Then the following statistic is *F*-distributed:

$$F = \frac{T-N}{m}\left(\frac{SSR_R}{SSR} - 1\right) \sim F(m, T-N) \tag{57}$$

We calculate this statistic for the hypotheses $H_{0,1}$ to $H_{0,3}$.

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